JOURNAL OF APPROXIMATION THEORY **62**, 192–196 (1990)

A Result on the Imaginary Zeros of $J_\nu''(z)$

E. K. IFANTIS AND P. D. SIAFARIKAS

Department of Mathematics, University of Patras, 26510 Patras, Greece

Communicated by Paul Nevai

Received July 14, 1988; revised December 1, 1988

The positive function $\rho(\nu)$, where $\pm i\rho(\nu)$ are the imaginary zeros of the second derivative of the Bessel function $J_\nu(z)$ of the first kind and order $\nu > -1$, increases for $0 < \nu \leqslant 0.4526$ and decreases for $0.5 \leqslant \nu < 1$. This is in full agreement with the numerical results given by M. K. Kerimov and S. L. Skorokhodov (*U.S.S.R. Comput. Math. Math. Phys.* **25**, No. 6 (1985), 101–107).

1. INTRODUCTION

This paper is concerned with the monotonicity of the positive function $\rho(\nu)$, where $\pm i\rho(\nu)$ are the imaginary zeros of the second derivative of the Bessel function $J_\nu(z)$ of the first kind and order $\nu > -1$. In [4] the existence of such a unique pair $\pm i\rho(\nu)$ of purely imaginary zeros of $J_\nu''(z)$ has been proved in the case $0 < \nu < 1$. In that case, on the basis of numerical calculations, M. K. Kerimov and S. L. Skorokhodov note in [6, p. 103] that one of the zeros of $J_\nu''(z)$ moves along the imaginary axis, reaches its maximum distance from the origin of coordinates roughly at $\nu = 0.5$, and then returns to the beginning. In this paper we give an analytical proof of this numerical observation; namely, there is a value $\nu_0 \in [0.4526, 0.5]$ such that $\rho(\nu)$ is strictly increasing on $[0, \nu_0]$ and decreasing on $[\nu_0, 1]$.

2. MAIN RESULT

First of all we prove that any zero $i\rho(\nu)$ of the function $M_\nu(z) = (\beta z^2 + \alpha(\nu))\, J_\nu(z) + zJ_\nu'(z)$, where $\alpha(\nu)$ is a differentiable function and $J_\nu'(z)$ is the derivative of $J_\nu(z)$, satisfies a differential equation of the form

$$\frac{d\rho^2(\nu)}{d\nu} = \frac{\rho^2(\nu)(1 + \alpha'(\nu)) + u(\nu)}{\nu + \alpha(\nu) + \varphi(\nu)}, \tag{2.0}$$

0021-9045/90 $3.00

where $u(\nu)$ is a negative function for every $\nu > -1$ and $\varphi(\nu)$ is a positive function for every $\nu > -1$.

In the particular case where $\beta = 1$ and $\alpha(\nu) = -\nu^2$ the zeros of $M_\nu(z)$ are the same as those of $J_\nu''(z)$. Let $i\rho(\nu)$ be an imaginary zero of $M_\nu(z)$, i.e., $[-\beta\rho^2(\nu) + \alpha(\nu)]\, J_\nu(i\rho(\nu)) + i\rho(\nu)\, J_\nu'(i\rho(\nu)) = 0$. Using the well known recurrence relation [8, 45] $i\rho(\nu)\, J_\nu'(i\rho(\nu)) = \nu J_\nu(i\rho(\nu)) - i\rho(\nu)\, J_{\nu+1}(i\rho(\nu))$, and the Mittag-Leffler expansion [8, 497]

$$\frac{J_{\nu+1}(i\rho(\nu))}{J_\nu(i\rho(\nu))} = 2i\rho(\nu) \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2 + \rho^2(\nu)}, \qquad \nu > -1$$

we find that any imaginary zero of $M_\nu(z)$ satisfies the equation

$$\beta - \frac{\alpha(\nu) + \nu}{\rho^2(\nu)} = 2 \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2 + \rho^2(\nu)}, \tag{2.1}$$

where $j_{\nu,n}$ is the nth positive zero of $J_\nu(z)$. Differentiation of both sides of (2.1) gives the differential equation (2.0) with

$$u(\nu) = -4\rho^4(\nu) \sum_{n=1}^{\infty} \frac{j_{\nu,n}(dj_{\nu,n}/d\nu)}{(j_{\nu,n}^2 + \rho^2(\nu))^2} < 0, \qquad \text{for every} \quad \nu > -1$$

and

$$\varphi(\nu) = 2\rho^4(\nu) \sum_{n=1}^{\infty} \frac{1}{(j_{\nu,n}^2 + \rho^2(\nu))^2} > 0, \qquad \text{for every} \quad \nu > -1.$$

For $\alpha(\nu) = -\nu^2$ the differential equation (2.0) takes the form

$$\lambda(\nu) \frac{d\rho^2(\nu)}{d\nu} = \mu(\nu), \tag{2.2}$$

where

$$\lambda(\nu) = \frac{\nu(1-\nu)}{\rho^4(\nu)} + 2 \sum_{n=1}^{\infty} \frac{1}{(j_{\nu,n}^2 + \rho^2(\nu)} > 0, \qquad \text{for} \quad 0 < \nu < 1$$

and

$$\mu(\nu) = \frac{1-2\nu}{\rho^2(\nu)} - 4 \sum_{n=1}^{\infty} \frac{j_{\nu,n}(dj_{\nu,n}/d\nu)}{(j_{\nu,n}^2 + \rho^2)^2} < 0, \qquad \text{for} \quad \frac{1}{2} \leqslant \nu < 1. \tag{2.3}$$

This shows that

$$\frac{d\rho^2(\nu)}{d\nu} < 0, \qquad \text{for} \quad \frac{1}{2} \leqslant \nu < 1.$$

We examine the function $\mu(\nu)$ in $(0, \frac{1}{2})$. Using the inequalities [4, 5]

$$\rho^2(\nu) \leqslant \frac{2\nu(1-\nu^2)}{2\nu+1}, \qquad 0<\nu<1$$
$$j_{\nu,n}\frac{dj_{\nu,n}}{d\nu} < \frac{1}{\nu+4}\left[j_{\nu,n}^2 + 8 - \frac{16(\nu+1)^2}{j_{\nu,n}^2} + \frac{32(\nu+1)^2(\nu+2)^2}{j_{\nu,n}^4}\right], \quad \nu>-1 \tag{2.4}$$

and the Rayleigh sums

$$\sigma_\nu^{(m)} = \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^{2m}},$$

obtainable from the formulas

$$\sigma_\nu^{(1)} = \frac{1}{4(\nu+1)}, \qquad (\nu+n)\sigma_\nu^{(n)} = \sum_{k=1}^{n-1} \sigma_\nu^{(k)}\sigma_\nu^{(n-k)}, \qquad n=2,3,..., \text{ [7]}$$

we have from (2.3)

$$\begin{aligned}\mu(\nu) &= \frac{1-2\nu}{\rho^2(\nu)} - 4\sum_{n=1}^{\infty}\frac{j_{\nu,n}(dj_{\nu,n}/d\nu)}{(j_{\nu,n}^2+\rho^2(\nu))^2} \\ &> \frac{(1-2\nu)(1+2\nu)}{2\nu(1-\nu^2)} - \frac{4}{\nu+4}\sigma_\nu^{(1)} - \frac{32}{\nu+4}\sigma_\nu^{(2)} \\ &\quad + \frac{2^6(\nu+1)^2}{\nu+4}\sigma_\nu^{(3)} - \frac{2^7(\nu+1)^2(\nu+2)^2}{\nu+4}\sigma_\nu^{(4)} \\ &= \frac{1}{1+\nu}\left[\frac{1-4\nu^2}{2\nu(1-\nu)} - \frac{1}{\nu+4} - \frac{5\nu^2+29\nu+54}{2(\nu+1)(\nu+2)(\nu+3)(\nu+4)^2}\right] > 0, \\ &\qquad \text{for} \quad 0<\nu\leqslant 0.4526.\end{aligned}$$

So for $0<\nu\leqslant 0.4526$, it follows from (2.2) that $d\rho^2(\nu)/d\nu>0$.

The function $\rho^2(\nu)$ attains only one local maximum in the interval (0.4526, 0.5). In fact assume that $d\rho^2(\nu)/d\nu=0$ for some $\nu\in(0.4526, 0.5)$. We prove that $d^2\rho^2(\nu)/d\nu^2<0$ for this ν. Differentiation of (2.2) with respect to ν gives

$$\lambda(\nu)\frac{d^2\rho^2(\nu)}{d\nu^2} = \mu'(\nu), \tag{2.5}$$

where

$$\mu'(\nu) = -\frac{2}{\rho^2(\nu)} + 4 \sum_{n=1}^{\infty} \frac{4 j_{\nu,n}'^2 j_{\nu,n}^2 - (j_{\nu,n}'^2 + j_{\nu,n} j_{\nu,n}'')(j_{\nu,n}^2 + \rho^2)}{(j_{\nu,n}^2 + \rho^2)^3}$$

$$= -\frac{2}{\rho^2(\nu)} - 4 \sum_{n=1}^{\infty} \frac{j_{\nu,n}'^2}{(j_{\nu,n}^2 + \rho^2)^2}$$

$$+ 16 \sum_{n=1}^{\infty} \frac{j_{\nu,n}'^2 j_{\nu,n}^2}{(j_{\nu,n}^2 + \rho^2)^3} - 4 \sum_{n=1}^{\infty} \frac{j_{\nu,n} j_{\nu,n}''}{(j_{\nu,n}^2 + \rho^2)^2}. \tag{2.6}$$

Using the inequalities (2.4) and $j_{\nu,n} j_{\nu,n}'' > -j_{\nu,n}'$, $\nu \geqslant 0$ [2], we have from (2.6) that

$$\mu'(\nu) < -4 \sum_{n=1}^{\infty} \frac{j_{\nu,n}'^2}{(j_{\nu,n}^2 + \rho^2)^2} - \frac{2\nu + 1}{\nu(1 - \nu^2)}$$

$$+ 16 \sum_{n=1}^{\infty} \frac{j_{\nu,n}'^2}{j_{\nu,n}^4} + 4 \sum_{n=1}^{\infty} \frac{j_{\nu,n}'}{j_{\nu,n}^4}. \tag{2.7}$$

Since $j_{\nu,n}' < j_{\nu,n}/(\nu + 1)$, $\nu > -1$, and $j_{\nu,1} > \nu + 1$ [1, 3], we obtain from (2.7)

$$\mu'(\nu) < -4 \sum_{n=1}^{\infty} \frac{j_{\nu,n}'^2}{(j_{\nu,n}^2 + \rho^2)^2} - \frac{2\nu + 1}{\nu(1 - \nu^2)}$$

$$\frac{16}{(\nu + 1)^2} \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} + \frac{4}{\nu + 1} \frac{1}{j_{\nu,1}} \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2},$$

$$\mu'(\nu) < -4 \sum_{n=1}^{\infty} \frac{j_{\nu,n}'^2}{(j_{\nu,n}^2 + \rho^2)^2} - \frac{2\nu^3 + 10\nu^2 - \nu + 1}{(\nu + 1)^3 \nu(1 - \nu)} < 0,$$

$$\text{for} \quad \nu \in (0.4526, 0.5). \tag{2.8}$$

So from (2.5) and (2.8) it follows that $d^2\rho^2(\nu)/d\nu^2 < 0$, for $\nu \in (0.4526, 0.5)$.

Acknowledgment

The authors express their thanks to Professor A'. Elbert for helpful remarks and suggestions.

References

1. A'. Elbert, Concavity of the zeros of Bessel functions, *Stud. Sci. Math. Hungar.* **12** (1977), 81–88.
2. A'. Elbert and A. Laforgia, Some consequences of a lower bound for the second derivative of the zeros of Bessel functions, *J. Math. Anal. Appl.* **125** (1987), 1–5.

3. E. K. IFANTIS AND P. D. SIAFARIKAS, A differential equation for the zeros of Bessel functions, *Appl. Anal.* **20** (1985), 269–281.
4. E. K. IFANTIS, P. D. SIAFARIKAS, AND C. B. KOURIS, The imaginary zeros of a mixed Bessel function, *J. Appl. Math. Phys. (ZAMP)* **39** (1988), 157–165.
5. E. K. IFANTIS AND P. D. SIAFARIKAS, Differential inequalities for the positive zeros of Bessel functions, *J. Comput. Appl. Math.* **30** (1990).
6. M. K. KERIMOV AND S. L. SKOROKHODOV, Calculation of the multiple zeros of the derivatives of the cylindrical Bessel functions $J_\nu(z)$ and $Y_\nu(z)$, *U.S.S.R. Comput. Math. Math. Phys.* **25**, No. 6 (1985), 101–107.
7. E. C. OBI, A note on the Rayleigh polynomials, *SIAM J. Math. Anal.* **9** (1978), 825–834.
8. G. N. WATSON, "A Treatise on the Theory of Bessel Functions," Cambridge Univ. Press, Cambridge, 1965.